October Math Gems

Problem of the week 10

§1 Problems

Problem 1.1. Let a, b be non-negative real numbers such that $a + b = 1$. Prove the following inequality is true

$$\frac{a+1}{b+2} + \frac{b+1}{a+2} \leq \frac{4}{3}$$

Problem 1.2. Let a, b be non-negative real numbers such that $a + b = 2$. Prove the following inequality:

$$\sqrt{a^2 + b + 2} + \sqrt{b^2 + a + 2} \geq 4$$

Problem 1.3. Let $a, b > 0$ and that $a^2 + b^2 = 2$. Prove that

$$\frac{2a^2}{b} + \frac{3b}{a} \geq 5$$

Problem 1.4. Let a, b be two real numbers. Prove that

$$(a+b)^2 \geq 4ab$$

Problem 1.5. Simplify

$$\frac{2^{54} + 1}{2^{27} + 2^{14} + 1}$$

Problem 1.6. Given that

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

Find $a + 2b + 3c$

Problem 1.7. Find the solutions (x, y) to the equations

$$\begin{cases} x^4 + 2x^3 - y = \sqrt{3} - \frac{1}{4} \\ y^4 + 2y^3 - x = -\sqrt{3} - \frac{1}{4} \end{cases}$$

Problem 1.8. Determine which number is bigger, $99!$ or 50^{99}

Problem 1.9. Given that

$$x^3 + 3x^2 + 5x - 17 = 0$$
$$y^3 - 3y^2 + 5y + 11 = 0$$

Find $x + y$

Problem 1.10. Given that

$$2\cos 40^\circ \sin\theta = sin(160 - \theta)$$

Solve for θ.

Problem 1.11. The number of positive integral values n for which $(n^3 - 8n^2 + 20n - 13)$ is a prime is ?

Problem 1.12. What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integer n ?

Problem 1.13. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and $110n^3$. The number $81n^4$ have D positive integer divisors. what is the value of $\frac{D}{5}$?

Problem 1.14. Given that

$$x = \lfloor \sqrt[3]{1} \rfloor + \lfloor \sqrt[3]{2} \rfloor + \lfloor \sqrt[3]{3} \rfloor + \lfloor \sqrt[3]{4} \rfloor + \lfloor \sqrt[3]{5} \rfloor + \cdots + \lfloor \sqrt[3]{7999} \rfloor$$

find the value of $\lfloor \frac{x}{5000} \rfloor$, where $\lfloor y \rfloor$ denotes to the greatest integer function less than or equal to y.

Problem 1.15. How many digits has the number $9^{30}4^{71}$?

Problem 1.16. If x is a real number that satisfies

$$\lfloor x + \frac{11}{100} \rfloor + \lfloor x + \frac{12}{100} \rfloor + \lfloor x + \frac{13}{100} \rfloor + \lfloor x + \frac{14}{100} \rfloor + \cdots + \lfloor x + \frac{99}{100} \rfloor = 765$$

find the value of $900 - \lfloor 100x \rfloor$.

Problem 1.17. For any real number x, let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x \rfloor$ denotes to the greatest integer function less than or equal to x. Find the value of

$$2010 - \sum_{k=1}^{2010} \lceil \frac{2010}{k} - \lfloor \frac{2100}{k} \rfloor \rceil$$

Problem 1.18. Given $x + y = \sqrt{3\sqrt{5} - \sqrt{2}}$ and $x - y = \sqrt{3\sqrt{2} - \sqrt{5}}$. What is the value of xy?

Problem 1.19. Evaluate x in the simplest form then find the sum of all digits of x. Where x is given as

$$x = \sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{\cdots}}}}}$$

Problem 1.20. Find the number of ordered pairs of positive integers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0$$
